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ROTATION MEANS OF PROJECTIONS

BY
ERWIN LUTWAK
Department of Mathematics,
Polytechnic University, Brooklyn, NY 11201, USA

ABSTRACT
An inequality is obtained between the Quermassintegrals of a convex body and power means of the Quermassintegrals of projections of the body onto subspaces. This inequality is shown to be a strengthened form of the classical inequality between the Quermassintegrals of a convex body. It is used to derive inequalities for rotation means of products of lower dimensional Quermassintegrals of convex bodies, which generalize inequalities obtained by Chakerian, Heil, Knothe, Schneider, and others.

Introduction

Let $K_1, \ldots, K_p$ be convex bodies (compact convex sets with nonempty interiors) in Euclidean n-space, $\mathbf{R}^n$, $n \geqq 2$. We consider p fixed subspaces of $\mathbf{R}^n, \xi_1, \ldots, \xi_p$, and let $n_i = \dim(\xi_i)$. Let $\mathrm{SO}(n)$ denote the group of proper rotations (acting on S^{n-1}) with Haar measure μ, normalized so that $\mu(\mathrm{SO}(n)) = 1$. For $g \in \mathrm{SO}(n)$ we denote by $g\xi_i$ the image of ξ_i under g, and the image of the orthogonal projection of K_i onto $g\xi_i$ will be written $K_i \mid g\xi_i$. If j_i is a nonnegative integer less than n_i, then we use $W_{j_i}(K_i \mid g\xi_i)$ to denote the j_i-th n_i-dimensional Quermassintegral (see [1], [10], [14], or [25] for definitions) of the body $K_i \mid g\xi_i$ in the space $g\xi_i$. Thus, for example, $W_0(K_i \mid g\xi_i)$ is the n_i-dimensional volume of $K_i \mid g\xi_i$, while $W_1(K_i \mid g\xi_i)$ is $(1/n_i)$ times the surface area of $K_i \mid g\xi_i$ computed in the space $g\xi_i$. To simplify our notation we will use $\boldsymbol{K}, \boldsymbol{\xi}, \boldsymbol{j}, \boldsymbol{n}$ to denote the p-tuples $(K_1, \ldots, K_p)$, $(\xi_1, \ldots, \xi_p)$, $(j_1, \ldots, j_p)$, and $(n_1, \ldots, n_p)$. Statements about p-tuples such as $\boldsymbol{K} = K$ ($\boldsymbol{K}$ is constant) and $\boldsymbol{j} \leqq j$ should be read $K_1 = \cdots = K_p = K$ and $j_i \leqq j$, for all i. We call the p-tuple $\boldsymbol{\xi}$ orthogonal if each of its entries is orthogonal to the other entries. We observe that $\boldsymbol{\xi}$ determines $\boldsymbol{n}$; i.e., $\boldsymbol{n} = \dim(\boldsymbol{\xi})$.

In this note we shall be concerned with means (over $\mathrm{SO}(n)$) of products of

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Quermassintegrals of the $K_i \mid g\xi_i$. Specifically, given p-tuples $\boldsymbol{K} = (K_1, \ldots, K_p)$, $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_p)$, and $\boldsymbol{j} = (j_1, \ldots, j_p)$, such that $0 \leqq \boldsymbol{j} < \boldsymbol{n}$ $(0 \leqq j_i < n_i)$, we define the mean $J_p(\boldsymbol{K}, \boldsymbol{\xi}, \boldsymbol{j})$ by

$$J_p(\boldsymbol{K}, \boldsymbol{\xi}, \boldsymbol{j}) = \int_{SO(n)} W_{j_1}(K_1 \mid g\xi_1) \cdots W_{j_p}(K_p \mid g\xi_p) d\mu(g).$$

For $\boldsymbol{K} = K$, and various choices of $\boldsymbol{\xi}$, $\boldsymbol{j}$, and p, one can interpret $J_p(\boldsymbol{K}, \boldsymbol{\xi}, \boldsymbol{j})$ as the mean of certain Quermassintegrals of various bodies circumscribed about K. See Chakerian [4] for a discussion of this. For $p = 1$, $J_p(\boldsymbol{K}, \boldsymbol{\xi}, \boldsymbol{j})$ is proportional to one of the n-dimensional Quermassintegrals of K. See, for example, [10, p. 232].

A number of investigators have considered the problem of finding inequalities that give sharp lower bounds for $J_p(\boldsymbol{K}, \boldsymbol{\xi}, \boldsymbol{j})$ in terms of the n-dimensional Quermassintegrals of the K_i, under a variety of restrictions on $\boldsymbol{K}$, $\boldsymbol{\xi}$, $\boldsymbol{j}$, $\boldsymbol{n}$ and p.

The problem for $\boldsymbol{n} = 1$ ($\boldsymbol{j} = 0$ is the only possibility when $\boldsymbol{n} = 1$) in the plane ($n = 2$), was considered by Radziszewski [21] and Chernoff [8], for the case where $p = 2$, $\boldsymbol{K}$ is constant, and $\boldsymbol{\xi}$ is orthogonal. Heil [12] generalizes the inequality of Radziszewski and Chernoff by removing the restriction on $\boldsymbol{\xi}$. Extensions of Heil's inequality, without restrictions on $\boldsymbol{K}$, can be found in the works of Chakerian [6] and Chakerian and Sangwine-Yager [7].

The problem for $\boldsymbol{n} = 1$, in n-space, is considered by Schneider in [26]. He obtains an inequality for J_p for the case where $p = 2$, $\boldsymbol{K}$ is constant, and $\boldsymbol{\xi}$ is orthogonal. Chakerian [5] (see also [25, p. 232]) drops the restriction that $\boldsymbol{\xi}$ be orthogonal and gives a generalization of Schneider's inequality for all p. A similar problem (for $p \leqq n$) is considered in [16] (see also [2, pp. 170–171]), where Chakerian's condition that $\boldsymbol{K}$ be constant is replaced with the requirement that $\boldsymbol{\xi}$ be constant.

In [13], Knothe obtains inequalities for J_p, in 3-space, for the case where $p = 2$, $\boldsymbol{K}$ is constant, $\boldsymbol{\xi}$ is orthogonal, and $\boldsymbol{n} = (1, 2)$. A striking generalization of Knothe's results (and of some of the previously mentioned ones) was obtained by Chakerian in [4]. He obtains inequalities for J_p, in n-space, for the case where $\boldsymbol{K}$ is constant, $\boldsymbol{\xi}$ is orthogonal, and $\boldsymbol{n}$ is such that $n_1 + \cdots + n_p = n$.

It is shown in [17] that the Petty projection inequality can be used to obtain inequalities for J_p, for the case where $p \leqq n$, $\boldsymbol{\xi}$ is constant, $\boldsymbol{n} = n - 1$, and $\boldsymbol{j} = 0$. The same case, but only requiring $\boldsymbol{j}$ to be constant (rather than 0), is treated in [18]. Stronger inequalities for J_p are then given in [19], where, in addition, the restriction on $\boldsymbol{j}$ is eliminated.

Interesting inequalities for means closely related to J_p are obtained by Sangwine-Yager in [23, 24].

It is easy to see that, for $n = 2$, an inequality for $J_p(\boldsymbol{K}, \boldsymbol{\xi}, \boldsymbol{j})$, involving the Quermassintegrals of the K_i, for the case where $\boldsymbol{\xi}$ is constant and $\boldsymbol{K}$ is arbitrary, will quickly yield such an inequality for arbitrary $\boldsymbol{\xi}$ and $\boldsymbol{K}$. This follows if one uses the commutativity of SO(2). For $n > 2$, it is not clear if an inequality for $J_p(\boldsymbol{K}, \boldsymbol{\xi}, \boldsymbol{j})$, for constant $\boldsymbol{\xi}$, can somehow be used to obtain a similar inequality for arbitrary $\boldsymbol{\xi}$. This was brought to the author's attention by Professor G. D. Chakerian.

The purpose of this note is to derive inequalities for $J_p(\boldsymbol{K}, \boldsymbol{\xi}, \boldsymbol{j})$, without restrictions on $\boldsymbol{n}$, $\boldsymbol{K}$, $\boldsymbol{\xi}$, $\boldsymbol{j}$, or p.

We shall use $W_j(K)$ to denote the j-th n-dimensional Quermassintegral of a convex body K in $\mathbf{R}^n$. Rather than $W_0(K)$, we write $V(K)$ for the volume of K. For the unit sphere in $\mathbf{R}^n$ we write S^{n-1}, for the unit ball in $\mathbf{R}^n$ we use B, and for the n-dimensional volume of B we write ω_n.

Our main result is:

THEOREM 1. *If $K_1, \ldots, K_p$ are convex bodies in $\mathbf{R}^n$, $\xi_1, \ldots, \xi_p$ are fixed subspaces of $\mathbf{R}^n$ of dimensions $n_1, \ldots, n_p$, respectively, and $j_1, \ldots, j_p$ are nonnegative integers such that $j_i < n_i$, then*

$$(\omega_{n_1} \cdots \omega_{n_p}) \prod_{i=1}^{p} [W_{j_i}(K_i)/\omega_n]^{(n_i - j_i)/(n - j_i)} \leqq J_p(\boldsymbol{K}, \boldsymbol{\xi}, \boldsymbol{j}),$$

with equality if and only if the K_i are balls.

Projection means of Quermassintegrals

In order to prove Theorem 1 we shall establish inequalities (Lemma 1) relating various power means of the Quermassintegrals of projections of a convex body over different Grassmann manifolds of subspaces.

For $1 \leqq i < n$, let $G(n, i)$ denote the Grassmann manifold of i-dimensional subspaces of $\mathbf{R}^n$. The (rotation invariant) Haar measure on $G(n, i)$ will be denoted by μ_i, and we assume it to be normalized so that $\mu_i(G(n, i)) = 1$. We recall that μ_{n-1} is essentially a constant multiple of spherical Lebesgue measure on a hemisphere of S^{n-1} (see, for example, [25, pp. 216–217]). If $\xi \in G(n, i)$, then, for $j < i$, we use $G(\xi, j)$ to denote the space of j-dimensional subspaces of $\mathbf{R}^n$ which are contained in ξ. For the Haar measure on $G(\xi, j)$ we use $\mu_j(\xi; \cdot)$, and we assume that it is normalized so that $\mu_j(\xi; G(\xi, j)) = 1$.

If K is a convex body in $\mathbf{R}^n$, $\xi \in G(n, i)$, and q is a nonnegative integer less than i, then $W_q(K \mid \xi)$ will denote the q-th i-dimensional Quermassintegral of $K \mid \xi$ (computed in ξ). If r is a nonzero real number, we define the r-th power mean of $W_q(K\,|\,\cdot)/\omega_i$ over $G(n,i)$, $M_r[W_q(K\,|\,\cdot);\ G(n,i)]$, by

$$M_r[W_q(K\,|\,\cdot);\ G(n,i)]=\left[\int_{G(n,i)}[W_q(K\,|\,\xi)/\omega_i]^r d\mu_i(\xi)\right]^{1/r}.$$

For $r=-\infty$, 0, or ∞, we can define $M_r[W_q(K\,|\,\cdot);\ G(n,i)]$ by

$$M_r[W_q(K\,|\,\cdot);\ G(n,i)]=\lim_{s\to r}M_s[W_q(K\,|\,\cdot);\ G(n,i)].$$

Note that our normalization is chosen so that for the unit ball we have

$$M_r[W_q(B\,|\,\cdot);\ G(n,i)]=1,$$

for all i,q,r. If L is a convex body in $\xi\in G(n,i)$, then we use $M_r[W_q(L\,|\,\cdot);\ G(\xi,j)]$ to denote the r-th power mean of the q-th j-dimensional Quermassintegral $W_q(L\,|\,\cdot)/\omega_j$ over $G(\xi,j)$.

We shall often make use of the fact that, unless $W_q(K\,|\,\cdot)$ is constant on $G(n,i)$, the power means $M_r[W_q(K\,|\,\cdot);\ G(n,i)]$ are strictly increasing in r. This is an immediate consequence of the Hölder (or Jensen) inequality (see, for example, [9, p. 88]).

We shall use S to denote spherical Lebesgue measure on the unit sphere S^{n-1}. For $u\in S^{n-1}$, we use E_u to denote the hyperplane orthogonal to u.

A version (see [17]) of the Petty projection inequality [20] (see [18] for an alternate proof of the Petty projection inequality) states that for a convex body K in $\mathbf{R}^n$,

$$\omega_{n-1}(V(K)/\omega_n)^{(n-1)/n}\leqq\left[\frac{1}{n\omega_n}\int_{S^{n-1}}V(K\,|\,E_u)^{-n}dS(u)\right]^{-1/n},$$

with equality if and only if K is an ellipsoid. An extension of this inequality [19, Corollary 5.18] states that for a convex body K in $\mathbf{R}^n$, and an integer q such that $0<q<n-1$,

$$\omega_{n-1}(W_q(K)/\omega_n)^{(n-q-1)/(n-q)}\leqq\left[\frac{1}{n\omega_n}\int_{S^{n-1}}W_q(K\,|\,E_u)^{-n}dS(u)\right]^{-1/n},$$

with equality (for $0<q<n-1$) if and only if K is a ball. It will be convenient to combine and rewrite these results:

If K is a convex body in $\mathbf{R}^n$ and q is a nonnegative integer less than $n-1$, then

$$[W_q(K)/\omega_n]^{n-q-1}\leqq M_{-n}[W_q(K\,|\,\cdot);\ G(n,n-1)]^{n-q}, \tag{1}$$

with equality if and only if, for $q=0$, K is an ellipsoid, and, for $q>0$, K is a ball.

From this inequality we shall obtain:

LEMMA 1. *If K is a convex body in $\mathbf{R}^n$, and i, j, q are integers such that $0 \leqq q < j < i \leqq n-1$, then*

$$M_{-i}[W_q(K \mid \cdot);\ G(n,i)]^{j-q} \leqq M_{-j}[W_q(K \mid \cdot);\ G(n,j)]^{i-q},$$

with equality if and only if K is a ball.

PROOF. We first consider the case where $j = i-1$. Suppose $\xi \in G(n,i)$. Inequality (1) for the convex body $K \mid \xi$ in the i-dimensional space ξ tells us that

$$[W_q(K \mid \xi)/\omega_i]^{i-q-1} \leqq M_{-i}[W_q((K \mid \xi) \mid \cdot);\ G(\xi, i-1)]^{i-q},$$

with equality if and only if, for $q = 0$, $K \mid \xi$ is an ellipsoid, and, for $q > 0$, $K \mid \xi$ is a ball. If ζ is a subspace of $\mathbf{R}^n$ which is contained in ξ, then $(K \mid \xi) \mid \zeta = K \mid \zeta$. We can therefore rewrite this inequality as

$$\int_{G(\xi,i-1)} [W_q(K \mid \zeta)/\omega_{i-1}]^{-i} d\mu_{i-1}(\xi;\zeta) \leqq [W_q(K \mid \xi)/\omega_i]^{-i(i-q-1)/(i-q)},$$

with equality if and only if, for $q = 0$, $K \mid \xi$ is an ellipsoid, and, for $q > 0$, $K \mid \xi$ is a ball. We integrate both sides over $G(n,i)$ and get

$$\int_{G(n,i)} \int_{G(\xi,i-1)} [W_q(K \mid \zeta)/\omega_{i-1}]^{-i} d\mu_{i-1}(\xi;\zeta) d\mu_i(\xi)$$
$$\leqq \int_{G(n,i)} [W_q(K \mid \xi)/\omega_i]^{-i(i-q-1)/(i-q)} d\mu_i(\xi),$$

with equality if and only if, for $q = 0$, K is an ellipsoid, and, for $q > 0$, K is a ball. The equality conditions, for $q = 0$, follow from the fact (see Chakerian [3, p. 20], or Rogers [22, p. 99]) that ellipsoids are characterized by the property that, for some $k > 1$, their k-dimensional projections are ellipsoids. A theorem of Rogers [22, Theorem 1] states that a pair of convex bodies are homothetic provided that, for some $k > 1$, their k-dimensional projections are homothetic. From this it follows that balls are characterized by the property that, for some $k > 1$, their k-dimensional projections are balls. The equality conditions, for $q > 0$, follow from this characterization of balls.

Since $M_r[W_q(K \mid \cdot);\ G(n, i-1)]$ is increasing in r, we have

$$M_{-(i-1)}[W_q(K \mid \cdot);\ G(n, i-1)]^{-i} \leqq \int_{G(n,i-1)} [W_q(K \mid \zeta)/\omega_{i-1}]^{-i} d\mu_{i-1}(\zeta)$$

with equality if and only if $W_q(K \mid \cdot)$ is constant on $G(n, i-1)$. But the quantity on the right in this inequality is the same as that on the left of the previous inequality. If we combine the two inequalities we get

$$M_{-(i-1)}[W_q(K\,|\,\cdot);\ G(n,i-1)]^{-i} \leqq \int_{G(n,i)} [W_q(K\,|\,\xi)/\omega_i]^{-i(i-q-1)/(i-q)} d\mu_i(\xi),$$

with equality if and only if K is a ball. Only the equality conditions for $q=0$ need justification. Note that for $q=0$, if K is a body for which equality occurs, then (it follows from the equality conditions of the two previous inequalities) the projection of K onto any i-dimensional subspace would, in that subspace, be a centrally symmetric body (an ellipsoid) of constant brightness and hence an i-dimensional ball.

Since $M_r[W_q(K\,|\,\cdot);\ G(n,i)]$ is increasing in r, we have

$$\int_{G(n,i)} [W_q(K\,|\,\xi)/\omega_i]^{-i(i-q-1)/(i-q)} d\mu_i(\xi) \leqq M_{-i}[W_q(K\,|\,\cdot);\ G(n,i)]^{-i(i-q-1)/(i-q)}.$$

If we combine the last two inequalities we get

$$M_{-i}[W_q(K\,|\,\cdot);\ G(n,i)]^{1/(i-q)} \leqq M_{-(i-1)}[W_q(K\,|\,\cdot);\ G(n,i-1)]^{1/(i-q-1)},$$

with equality if and only if K is a ball. This completes the proof for the case where $(j,i)=(i-1,i)$.

If we combine a string of such inequalities for the cases $(j,j+1)$, $(j+1,j+2)$, $\dots$, $(i-1,i)$, we obtain the desired inequality for arbitrary (i,j).

From the fact that $M_r[W_q(K\,|\,\cdot);\ G(n,n-1)]$ is increasing in r, it follows that

$$M_{-n}[W_q(K\,|\,\cdot);\ G(n,n-1)] \leqq M_{-(n-1)}[W_q(K\,|\,\cdot);\ G(n,n-1)], \tag{2}$$

with equality if and only if K has constant $(n-q-1)$-girth. If we combine this inequality with inequality (1), and with the inequality of Lemma 1 (for the case where $i=n-1$), the result is:

THEOREM 2. *If K is a convex body in $\mathbf{R}^n$, and j,q are integers such that $0\leqq q<j\leqq n-1$, then*

$$[W_q(K)/\omega_n]^{j-q} \leqq M_{-j}[W_q(K\,|\,\cdot);\ G(n,j)]^{n-q},$$

with equality if and only if K is a ball.

The case $j=n-1$ of Theorem 2 follows if we combine (1) and (2). There is no need, in this case, to appeal to Lemma 1.

We observe that the inequality of Theorem 2 is a 'stronger' inequality than the well-known inequality between the Quermassintegrals of a convex body. To see this note that

$$W_{n+q-i}(K) = \omega_n M_1[W_q(K \mid \cdot);\ G(n,i)]$$

(see, for example, [10, p. 232]), and since $M_r[W_q(K \mid \cdot);\ G(n,i)]$ is increasing in r, it follows that

$$\omega_n M_{-i}[W_q(K \mid \cdot);\ G(n,i)] \leqq W_{n+q-i}(K).$$

If we combine this with the inequality of Theorem 2 we obtain the classical inequality between the Quermassintegrals of a convex body:

If K is a convex body in $\mathbf{R}^n$ and q, i are such that $0 \leqq q < i < n$, then

$$\omega_n^{n-i} W_q(K)^{i-q} \leqq W_{n+q-i}(K)^{n-q},$$

with equality if and only if K is a ball.

Proof of Main Theorem

Let $\xi \in G(n,i)$ be a fixed i-dimensional subspace of $\mathbf{R}^n$, and let q be an integer such that $0 \leqq q < i$. It will be convenient to rewrite the inequality of Theorem 2 as

$$[W_q(K)/\omega_n]^{(i-q)/(n-q)} \leqq \left[\int_{\mathrm{SO}(n)} [W_q(K \mid g\xi)/\omega_i]^{-i} d\mu(g)\right]^{-1/i}, \tag{3}$$

with equality if and only if K is a ball.

We shall require the following consequence of the Hölder integral inequality:

LEMMA 2. *If $f_1, \ldots, f_p$ are positive continuous functions on $\mathrm{SO}(n)$, and $\alpha_1, \ldots, \alpha_p$ are positive real numbers, then*

$$\prod_{i=1}^{p}\left[\int_{\mathrm{SO}(n)} f_i(g)^{-\alpha_i} d\mu(g)\right]^{-1/\alpha_i} \leqq \int_{\mathrm{SO}(n)} f_1(g)\cdots f_p(g)\,d\mu(g),$$

with equality if and only if the f_i are constant.

PROOF. To prove this we use a form of the Hölder inequality [11, p. 140] (see also [9, p. 88]), which states that given positive measurable functions $F_0, F_1, \ldots, F_p$ on $\mathrm{SO}(n)$, and positive real numbers $\beta_0, \beta_1, \ldots, \beta_p$, whose reciprocals sum to unity, then

$$\int_{\mathrm{SO}(n)} F_0(g)\cdots F_p(g)\,d\mu(g) \leqq \prod_{i=0}^{p}\left[\int_{\mathrm{SO}(n)} F_i(g)^{\beta_i} d\mu(g)\right]^{1/\beta_i},$$

with equality if and only if there exist positive constants $c_1, \ldots, c_p$, such that $F_0^{\beta_0} = c_j F_j^{\beta_j}$ a.e. $[\mu]$.

To obtain the inequality of Lemma 2, for the case $p > 1$, we apply this version of the Hölder inequality with

$$\beta_0 = 1 + \sum_{i=1}^{p} \alpha_i^{-1},$$

and

$$F_0 = (f_1 \cdots f_p)^{1/\beta_0},$$

and, for $i > 0$,

$$\beta_i = \alpha_i \beta_0,$$

and

$$F_i = f_i^{-1/\beta_0}.$$

Equality in our inequality, implies the existence of positive constants $c_1, \ldots, c_p$, such that for all i, $f_1 \cdots f_p = c_i f_i^{-\alpha_i}$ a.e. $[\mu]$. Hence, there exist constants $c'_1, \ldots, c'_p$ such that for all i, $f_i = c'_i$ a.e. $[\mu]$. Thus, the open sets $D_i = \{g \in \mathrm{SO}(n) : f_i(g) \neq c'_i\}$ have measure zero. But, on a compact topological group, an open set of Haar measure zero must be the empty set. Hence, the f_i are constant.

The case $p = 1$ follows from the case $p = 2$ if we take $f_2 = 1$.

Suppose $K_1, \ldots, K_p$ are convex bodies in $\mathbf{R}^n$, $\xi_1, \ldots, \xi_p$ are fixed subspaces of $\mathbf{R}^n$ of dimensions $n_1, \ldots, n_p$, respectively, and $j_1, \ldots, j_p$ are nonnegative integers such that $j_i < n_i$. It follows from Lemma 2 that

$$\prod_{i=1}^{p} \left[\int_{\mathrm{SO}(n)} [W_{j_i}(K_i \mid g\xi_i)/\omega_{n_i}]^{-n_i} d\mu(g) \right]^{-1/n_i}$$
$$\leqq (\omega_{n_1} \cdots \omega_{n_p})^{-1} \int_{\mathrm{SO}(n)} W_{j_1}(K_1 \mid g\xi_1) \cdots W_{j_p}(K_p \mid g\xi_p) d\mu(g),$$

with equality if and only if $W_{j_i}(K_i \mid \cdot)$ is constant on $G(n, n_i)$. If we combine this with inequality (3) we get

$$(\omega_{n_1} \cdots \omega_{n_p}) \prod_{i=1}^{p} [W_{j_i}(K_i)/\omega_n]^{(n_i - j_i)/(n - j_i)} \leqq \int_{\mathrm{SO}(n)} W_{j_1}(K_1 \mid g\xi_1) \cdots W_{j_p}(K_p \mid g\xi_p) d\mu(g),$$

with equality if and only if the K_i are balls. This proves Theorem 1.

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